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ON SMOOTH LATTICE POLYTOPES WITH SMALL DEGREE

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ABSTRACT. Toric geometry provides a bridge between the theory of polytopes and algebraic geometry: one can associate to each lattice polytope a polarized toric variety. In this paper we explore this correspondence to classify smooth lattice polytopes having small degree, extending a classification provided by Dickenstein, Di Rocco and Piene. Our approach consists in interpreting the degree of a polytope as a geometric invariant of the corresponding polarized variety, and then applying techniques from Adjunction Theory and Mori Theory.

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1. INTRODUCTION

The *degree* of an n-dimensional lattice polytope $P \subset \mathbb{R}^n$ is the smallest non-negative integer d such that kP contains no interior lattice points for $1 \leq k \leq n-d$. The degree d of P is related to the *Ehrhart series* of P as follows. For each positive integer m, let $f_P(m)$ denote the number of lattice points in mP, and consider the Ehrhart series

$$F_P(t) := \sum_{m \geq 1} f_P(m) t^m.$$

It turns out that $h^*_P(t) := \frac{F_P(t)}{(1-t)^{n+1}}$ is a polynomial of degree d in t. (See [BR07] for more details on Ehrhart series and h^*-polynomials.) The *codegree* of P is defined as $\operatorname{codeg}(P) = n+1-d$. It is the smallest non-negative integer c such that cP contains an interior lattice point.

Lattice polytopes with small degree are very special. It is not difficult to see that lattice polytopes with degree $d = 0$ are precisely unimodular simplices ([BN07, Proposition 1.4]). In [BN07, Theorem 2.5], Batyrev and Nill classified lattice polytopes with degree $d = 1$. They all belong to a special class of lattice polytopes, called *Cayley polytopes*. A Cayley polytope is a lattice polytope affinely isomorphic to

$$P_0 * ... * P_k := \operatorname{Conv}\left(P_0 \times \{0\}, P_1 \times \{e_1\}, \cdots, P_k \times \{e_k\}\right) \subset \mathbb{R}^m \times \mathbb{R}^k,$$

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where the P_i's are m-dimensional lattice polytopes in $\mathbb{R}^m$, and $\{e_1, ..., e_k\}$ is a basis for $\mathbb{Z}^k$. Batyrev and Nill also posed the following problem: to find a function $N(d)$ such that every lattice polytope of degree d and dimension $n > N(d)$ is a Cayley polytope. In [HNP09, Theorem 1.2], Hasse, Nill and Payne solved this problem with the quadratic polynomial $N(d) = (d^2+19d-4)/2$. It was conjectured in [DN10, Conjecture 1.2] that one can take $N(d) = 2d$. This would be a sharp bound. Indeed, let Δ_n denote the standard n-dimensional unimodular simplex. If n is even, then $2\Delta_n$ has degree $d = \frac{n}{2}$, but it is not a Cayley polytope.

While the methods of [BN07] and [HNP09] are purely combinatorial, Hasse, Nill and Payne pointed out that these results can be interpreted in terms of Adjunction Theory on toric varieties. This point of view was then explored by Dickenstein, Di Rocco and Piene in [DDRP09] to study *smooth* lattice polytopes with small degree. Recall that an n-dimensional lattice polytope P is smooth if there are exactly n facets incident to each vertex of P, and the primitive inner normal vectors of these facets form a basis for $\mathbb{Z}^n$. This condition is equivalent to saying that the toric variety associated to P is smooth. One has the following classification of smooth n-dimensional lattice polytopes P with degree $d < \frac{n}{2}$ (or, equivalently, $\operatorname{codeg}(P) \geq \frac{n+3}{2}$).

Theorem 1 ([DDRP09, Theorem 1.12] and [DN10, Theorem 1.6])**.** *Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional lattice polytope. Then $\operatorname{codeg}(P) \geq \frac{n+3}{2}$ if and only if P is affinely isomorphic to a Cayley polytope $P_0 * ... * P_k$, where all the P_i's have the same normal fan, and $k > \frac{n}{2}$.*

Theorem 1 was first proved in [DDRP09] under the additional assumption that P is a $\mathbb{Q}$-*normal* polytope. (See Definition 7 for the notion of $\mathbb{Q}$-normality.) Then, using combinatorial methods, Dickenstein and Nill showed in [DN10] that the inequality $\operatorname{codeg}(P) \geq \frac{n+3}{2}$ implies that P is $\mathbb{Q}$-normal.

The aim of this paper is to extend this classification. We address smooth n-dimensional lattice polytopes P of degree $d < \frac{n}{2} + 1$ (or, equivalently, $\operatorname{codeg}(P) \geq \frac{n+1}{2}$). Not all such polytopes are Cayley polytopes, and we need the following generalization of the Cayley condition, introduced in [DDRP09].

Definition 2. Let $P_0, \dots, P_k \subset \mathbb{R}^m$ be m-dimensional lattice polytopes, and s a positive integer. Set

$$[P_0 * ... * P_k]^s \ := \ \operatorname{Conv}\left(P_0 \times \{0\}, P_1 \times \{se_1\}, \cdots, P_k \times \{se_k\}\right) \subset \mathbb{R}^m \times \mathbb{R}^k,$$

where $\{e_1, ..., e_k\}$ is a basis for $\mathbb{Z}^k$. A lattice polytope P is an s^{th} *order generalized Cayley polytope* if it is affinely isomorphic to a polytope $[P_0 * ... * P_k]^s$ as above. If all the P_i's have the same normal fan, we write $P = \operatorname{Cayley}^s(P_0, \dots, P_k)$, and say that P is *strict*.

The following is our main result:

Theorem 3. *Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional $\mathbb{Q}$-normal lattice polytope. Then $\operatorname{codeg}(P) \geq \frac{n+1}{2}$ if and only if P is affinelly isomorphic to one of the following polytopes:*

(i) $s\Delta_1$, $s \geq 1$ $(n = 1)$;
(ii) $3\Delta_3$ $(n = 3)$;
(iii) $2\Delta_n$;
(iv) $\operatorname{Cayley}^1(P_0, \dots, P_k)$, *where* $k \geq \frac{n-1}{2}$;
(v) $\operatorname{Cayley}^2(a_0\Delta_1, a_1\Delta_1, \dots, a_{n-1}\Delta_1)$, *where n is odd and the a_i's are congruent modulo* 2.

Corollary 4. *Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional $\mathbb{Q}$-normal lattice polytope. If $\operatorname{codeg}(P) \geq \frac{n+1}{2}$, then P is a strict generalized Cayley polytope.*

In Example 18, we describe a smooth n-dimensional lattice polytope $P \subset \mathbb{R}^n$ with $\operatorname{codeg}(P) = \frac{n+1}{2}$ which is not a generalized Cayley polytope. So one cannot drop the assumption of $\mathbb{Q}$-normality in Corollary 4.

Our proof of Theorem 3 follows the strategy of [DDRP09]: we interpret the degree of P as a geometric invariant of the corresponding polarized variety (X, L), and then apply techniques from Adjunction Theory and Mori Theory. This approach naturally leads to introducing more refined invariants of lattice polytopes, which are the polytope counterparts of important invariants of polarized varieties. In particular, we consider the $\mathbb{Q}$-*codegree* $\mathrm{codeg}_{\mathbb{Q}}(P)$ of P (see Definition 7). This is a rational number that carries information about the birational geometry of (X, L). For $\mathbb{Q}$-normal smooth lattice polytopes, it satisfies $\lceil \mathrm{codeg}_{\mathbb{Q}}(P) \rceil = \mathrm{codeg}(P)$.

The following is the polytope version of a conjecture by Beltrametti and Sommese.

Conjecture 5 ([BS95, 7.18])**.** *Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional lattice polytope. If $\mathrm{codeg}_{\mathbb{Q}}(P) > \frac{n+1}{2}$, then P is $\mathbb{Q}$-normal.*

Remark 6. If Conjecture 5 holds, then Theorem 3 and Proposition 17 imply that smooth lattice polytopes P with $\mathrm{codeg}_{\mathbb{Q}}(P) > \frac{n+1}{2}$ are those in (iv) with $k \geq \frac{n}{2}$. These have $\mathbb{Q}$-codegree $\geq \frac{n+2}{2}$. Hence, if Conjecture 5 holds, then the $\mathbb{Q}$-codegree of smooth lattice polytopes does not assume values in the interval $\left(\frac{n+1}{2}, \frac{n+2}{2}\right)$.

Notation and conventions. We mostly follow the notation of [Ful93] for toric geometry. Given a fan Σ in $\mathbb{R}^n$, we denote by X_Σ the corresponding toric variety. For any cone $\sigma \in \Sigma$, we denote by $V(\sigma)$ the T-invariant subvariety of X_Σ associated to σ. For each integer $m \in \{1, \dots, n\}$, we denote by $\Sigma(m)$ the set of m-dimensional cones of Σ. We identify $\Sigma(1)$ with the set of primitive vectors of the 1-dimensional cones of Σ. Given a polytope $P \subset \mathbb{R}^n$, we denote by Σ_P the normal fan of P.

By abuse of notation, we identify a vector bundle on a variety with its corresponding locally free sheaf of sections. Given a vector bundle $\mathcal{E}$ on a variety Y, we denote by $\mathbb{P}_Y(\mathcal{E})$ the Grothendieck projectivization $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$.

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2. Preliminaries

2.1. **Adjoint polytopes, nef value and $\mathbb{Q}$-codegree.** Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. For each $t \in \mathbb{R}_{\geq 0}$, we let $P^{(t)}$ be the (possibly empty) polytope obtained by moving each facet of P toward its inner normal direction by a "lattice distance" of t units. More precisely, if $\Sigma_P(1) = \{\eta_i\}_{i \in \{1,\dots,r\}}$, and P is given by facet presentation

$$P \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; \langle \eta_i, x \rangle \geq -a_i,\ 1 \leq i \leq r \right\},$$

then $P^{(t)}$ is given by

$$P^{(t)} \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; \langle \eta_i, x \rangle \geq -a_i + t,\ 1 \leq i \leq r \right\}.$$

These are called *adjoint polytopes* in [DRHNP11]. Set

$$\sigma(P) := sup \left\{ t \geq 0 \;\middle|\; P^{(t)} \neq \varnothing \right\}.$$

As we increase t from 0 to $\sigma(P)$, $P^{(t)}$ will change its combinatorial type at some critical values, the first one being

$$\lambda(P) := sup \left\{ t \geq 0 \;\middle|\; P^{(t)} \neq \varnothing \text{ and } \Sigma_t := \Sigma_{P^{(t)}} = \Sigma_P \right\} \leq \sigma(P).$$

By [DRHNP11, Lemma 1.13], $\lambda(P) > 0$ if and only if the normal fan of P is $\mathbb{Q}$-Gorenstein. This happens for instance when P is a smooth lattice polytope.

Definition 7. Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope.

We say that P is $\mathbb{Q}$*-normal* if $\sigma(P) = \lambda(P)$.

The $\mathbb{Q}$*-codegree* of P is

$$\operatorname{codeg}_{\mathbb{Q}}(P) \;:=\; \sigma(P)^{-1}.$$

Suppose that $\lambda(P) > 0$. Then the *nef value* of P is

$$\tau(P) \;:=\; \lambda(P)^{-1}.$$

Remark 8. Let P be a lattice polytope. Then $\operatorname{codeg}_{\mathbb{Q}}(P) \leq \tau(P)$. For any positive integer k, the interior lattice points of kP are precisely the lattice points of $(kP)^{(1)}$, and $(kP)^{(1)} \neq \varnothing$ if and only if $P^{(1/k)} \neq \varnothing$. Hence $\operatorname{codeg}(P) \geq \lceil \operatorname{codeg}_{\mathbb{Q}}(P) \rceil$. By [DDRP09, Lemma 2.4], for a smooth lattice polytope P, $\tau(P) > \operatorname{codeg}(P) - 1$. Therefore, for a $\mathbb{Q}$-normal smooth lattice polytope P we have

$$\lceil \operatorname{codeg}_{\mathbb{Q}}(P) \rceil = \operatorname{codeg}(P).$$

Remark 9. Let P be a lattice polytope, and (X, L) the corresponding polarized toric variety. When X is $\mathbb{Q}$-Gorenstein (i.e., some nonzero multiple of K_X is Cartier), the family of adjoint polytopes $\{P^{(t)}\}_{0 \leq t \leq \sigma(P)}$ is the polytope counterpart of the *Minimal Model Program with scaling*, established in [BCHM10]. The projective varieties $X_t = X_{\Sigma_t}$ that appear as we increase t from 0 to $\sigma(P)$ are precisely the varieties that appear in the Minimal Model Program for X with scaling of L. A precise statement and proof can be found in [Nob12].

2.2. **Adjunction Theory.** Let (X, L) be a smooth polarized variety. This means that X is a smooth projective variety, and L is an ample divisor on X. Divisors of the form $L + mK_X$, $m > 0$, are called *adjoint divisors*, and play an important role in classification of projective varieties. We refer to [BS95] for an overview of classical adjuntion theory.

We denote by $N^1(X)$ the (finite-dimensional) $\mathbb{R}$-vector space of $\mathbb{R}$-divisors on X modulo numerical equivalence. The *nef cone* of X is the closed convex cone $Nef(X) \subset N^1(X)$ generated by classes of nef divisors on X (i.e., divisors having nonnegative intersection with every curve of X). By Kleiman's ampleness criterion, a divisor on X is ample if and only if its class lies in the interior of $Nef(X)$. The *cone of pseudo-effective divisors* of X is the closed convex cone $\overline{Eff}(X) \subset N^1(X)$ generated by classes of effective divisors on X. By Kodaira's lemma, the class of a divisor D lies in the interior of $\overline{Eff}(X)$ if and only if the linear system $|kD|$ defines a generically finite map for k sufficiently large and divisible. In this case, such map is in fact birational, and we say that D is *big*.

The following are important invariants of the polarized variety (X, L). The *nef threshold* of (X, L) is

$$\lambda(X, L) = \sup\left\{t \geq 0 \;\middle|\; [L + tK_X] \in Nef(X)\right\}.$$

This is a rational number by Kawamata's Rationality Theorem (see [KM98, Theorem 3.5]). The *effective threshold* of (X, L) is

$$\sigma(X, L) = \sup\left\{t \geq 0 \;\middle|\; [L + tK_X] \in \overline{Eff}(X)\right\}.$$

It follows from [BCHM10] that this is also a rational number (see [Ara10, Theorem 5.2]).

10 (The toric case)**.** Next we specialize to the toric case. We refer to [Ful93] for details and proofs.

Let $X = X_\Sigma$ be a smooth projective n-dimensional toric variety, write $\Sigma(1) = \{\eta_i\}_{i \in \{1, \ldots, r\}}$, and let $L_i = V(\eta_i)$ be the T-invariant divisor associated to η_i. The classes of the L_i's span $N^1(X) \cong \operatorname{Pic}(X) \otimes \mathbb{R}$, and generate the cone $\overline{Eff}(X)$.

The canonical divisor of X can be written as $K_X = -\sum_{i=1}^{r} L_i$.

Let $D = \sum_{i=1}^{r} a_i L_i$ be an invariant $\mathbb{R}$-divisor on X. We associate to D the following (possibly empty) polytope:

$$P_D = \Big\{ x \in \mathbb{R}^n \ \Big| \ \langle \eta_i, x \rangle \geq -a_i,\ 1 \leq i \leq r \Big\}.$$

Geometric properties of the divisor D are reflected by combinatorial properties of the polytope P_D. For instance:

- D is ample if and only if $\Sigma_{P_D} = \Sigma$.
- D is big if and only if P_D is n-dimensional.
- $[D] \in \overline{Eff}(X)$ if and only if $P_D \neq \varnothing$.

The above equivalences allow us to reinterpret the nef value and $\mathbb{Q}$-codegree of a lattice polytope in terms of invariants of the associated polarized toric variety. Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional lattice polytope, and denote by (X, L) the associated polarized toric variety. Notice that the polytope associated to the adjoint $\mathbb{R}$-divisor $L + tK_X$ is precisely the adjoint polytope $P^{(t)}$. Therefore

$$\lambda(P) = \lambda(X, L) \quad \text{and} \quad \sigma(P) = \sigma(X, L).$$

Moreover, $\dim P^{(t)} = n$ for $0 \leq t < \sigma(P)$, and $\dim P^{(\sigma(P))} < n$ (see also [DRHNP11, Proposition 1.6]).

2.3. **Ingredients from Mori Theory.** Let X be a smooth projective variety. We denote by $N_1(X)$ the $\mathbb{R}$-vector space of 1-cycles on X with real coefficients modulo numerical equivalence. The *Mori cone* of X is the closed convex cone $\overline{NE}(X) \subset N_1(X)$ generated by classes of irreducible curves on X. Intersection product of divisors and curves makes $N^1(X)$ and $N_1(X)$ dual vector spaces, and $Nef(X) \subset N^1(X)$ and $\overline{NE}(X) \subset N_1(X)$ dual cones.

Let N be a face of $\overline{NE}(X)$. The *contraction of* N is a surjective morphism $\phi_N : X \to Y$ with connected fibers onto a normal variety satisfying the following condition: the class of an irreducible curve $C \subset X$ lies in N if and only if $\phi_N(C)$ is a point. Stein Factorization guarantees that if such contraction exists, it is unique up to isomorphism. By the Contraction Theorem, if K_X is negative on $N \setminus \{0\}$ (in which case we say that N is a *negative extremal face* of $\overline{NE}(X)$), then ϕ_N exists (see [KM98, Theorem 3.7]). More precisely, if D is any nef divisor such that $(D = 0) \cap \overline{NE}(X) = N$ and k is sufficiently large and divisible, then $\big|kD\big|$ defines the contraction of N (see [KM98, Theorem 3.3]).

Let L be an ample divisor on X, and set $\lambda := \lambda(X, L)$. The adjoint $\mathbb{Q}$-divisor $L + \lambda K_X$ is nef but not ample, and thus defines a negative extremal face N of the Mori cone $\overline{NE}(X)$. We call the contraction of N the *nef value morphism* of (X, L), and denote it by $\phi_L : X \to Y$. If follows from the discussion of Section 2.2 that $\dim(Y) < \dim(X)$ if and only if $\lambda(X, L) = \sigma(X, L)$.

Let R be a negative extremal ray of the Mori cone $\overline{NE}(X)$, and $\phi_R : X \to Y$ the contraction of R. The *lenght* of R is

$$\mathfrak{l}(R) := min\big\{ -K_X \cdot C \ \big| \ C \subset X \text{ rational curve contracted by } \phi_R \big\}.$$

It satisfies $\mathfrak{l}(R) \leq \dim(X) + 1$ (see [KM98, Theorem 3.7]). A rational curve $C \subset X$ such that $[C] \in R$ and $\mathfrak{l}(R) = -K_X \cdot C$ is called an *extremal curve*. Let $E_R \subset X$ be the exceptional locus of ϕ_R, i.e., the locus of points at which ϕ_R is not an isomorphism. The following inequality is due to Ionescu-Wiśniewski (see [BS95, Theorem 6.36]). Let E be an irreducible component of E_R, and F an irreducible component of a fiber of the restriction $\phi_R|_E$. Then

$$\dim(E) + \dim(F) \geq \dim(X) + \mathfrak{l}(R) - 1. \tag{1}$$

If $E_R = X$, i.e., $\dim(Y) < \dim(X)$, we say that ϕ_R is a *contraction of fiber type*. If $\dim E_R = \dim(X) - 1$, then ϕ_R is birational and E_R is a prime divisor. In this case we say that ϕ_R is a *divisorial contraction*.

The following result describes the nef value morphism of polarized varieties with small nef threshold. It follows immediately from [BSW92, Theorems 3.1.1 and 2.5].

Theorem 11. *Let (X, L) be a smooth polarized variety of dimension n with associated nef value morphism $\phi_L : X \to Y$. Suppose that $\tau := \lambda(X, L)^{-1} \geq \frac{n+1}{2}$ and $1 \leq \dim(Y) \leq n-1$. Then there exists a negative extremal ray $R \subset \overline{NE}(X)$ of length $\mathfrak{l}(R) = \tau$ whose associated contraction $\phi_R : X \to Z$ is of fiber type and factors ϕ_L.*

When X is toric, we will see below that the contraction ϕ_R provided by Theorem 11 is a $\mathbb{P}^{\tau-1}$-bundle over a smooth toric variety Z.

12 (The toric case)**.** We now specialize to the toric case. We refer to [Rei83] for details and proofs.

Let $X = X_\Sigma$ be a smooth projective n-dimensional toric variety. Then $\overline{NE}(X)$ is generated by T-invariant rational curves $V(\omega)$, $\omega \in \Sigma(n-1)$. In particular, it is a rational polyhedral cone. Moreover, any face N of $\overline{NE}(X)$ admits a contraction $\phi_N : X \to Y$ onto a toric variety Y.

Let R be an extremal ray of the Mori cone, $\phi_R : X \to Y$ the contraction of R, and $E_R \subset X$ its exceptional locus. Then the restriction $\phi_R|_{E_R}$ makes E_R a $\mathbb{P}^d$-bundle over an invariant smooth subvariety $Z \subset Y$.

Remark 13. Let $X = X_\Sigma$ be a smooth projective n-dimensional toric variety, and $C \subset X$ an invariant curve whose class generates an extremal ray of $\overline{NE}(X)$. We have seen above that $-K_X \cdot C \leq \dim(X) + 1$. If equality holds, then (1) implies that $E = F = X$. Thus X is isomorphic to a projective space. Hence, if $\dim(X) = n$ and $X \not\cong \mathbb{P}^n$, then $-K_X \cdot C \leq n$ for every invariant curve $C \subset X$ whose class generates an extremal ray of $\overline{NE}(X)$. In particular, if L is an $\mathbb{R}$-divisor on X such that $L \cdot C \geq n$ for every invariant curve C, then one of the following holds:

- $K_X + L$ is nef; or
- $X \cong \mathbb{P}^n$ and $[L] = t[H]$, where H is a hyperplane and $n \leq t < n+1$.

This observation generalizes [Mus02, Corollary 4.2] to $\mathbb{R}$-divisors.

2.4. **Fano manifolds with large index.** Let X be a smooth projective variety. We say that X is a *Fano manifold* if the anticanonical divisor $-K_X$ is ample. In this case we define the *index* of X as the largest integer r dividing $-K_X$ in $\mathrm{Pic}(X)$. Fano manifolds with large index are very special. In [Wiś91], Wiśniewski classified n-dimensional Fano manifolds with index $r \geq \frac{n+1}{2}$. They satisfy one of the following conditions:

(1) X has Picard number one;
(2) $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$;
(3) $X \simeq \mathbb{P}^{r-1} \times Q^r$, where Q^r is an r-dimensional smooth hyperquadric;
(4) $X \simeq \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$; or
(5) $X \simeq \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{r-1})$.

Notice that many of those are not toric. The only smooth projective toric varieties with Picard number one are projective spaces. The smooth hyperquadric Q^r is not toric if $r > 2$. Finally, if E is a vector bundle over a toric variety Z, then $\mathbb{P}_Z(E)$ is toric if and only if E is a direct sum of line bundles. In particular $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$ is not toric if $r > 1$.

3. Cayley Polytopes and Toric Fibrations

In this section we describe the geometry of polarized toric varieties associated to generalized Cayley polytopes. We start by fixing the notation to be used throughout this section.

Notation 14. Let k be a positive integer, and $P_0, \ldots, P_k \subset \mathbb{R}^m$ m-dimensional lattice polytopes having the same normal fan Σ. Let $Y = X_\Sigma$ be the corresponding projective m-dimensional toric variety, and D_j the ample T-invariant divisor on Y associated to P_j. More precisely, write $\Sigma(1) = \{\eta_i\}_{i\in\{1,\ldots,r\}}$, and let P_j be given by facet presentation:

$$P_j \ = \ \Big\{ x \in \mathbb{R}^m \ \Big| \ \langle \eta_i, x\rangle \geq -a_{ij},\ 1 \leq i \leq r \ \Big\}.$$

Let $L_i = V(\eta_i)$ be the T-invariant Weil divisor on Y associated to η_i. Then $D_j = \sum_{i=1}^{r} a_{ij} L_i$.

15 (Strict Cayley polytopes)**.** By [CCD97, Section 3], the polarized toric variety associated to the strict Cayley polytope $P_0 * \ldots * P_k$ is

$$(X, L) \ \cong \ \Big(\mathbb{P}_Y\big(\mathcal{O}(D_0) \oplus \cdots \oplus \mathcal{O}(D_k)\big),\ \xi\Big),$$

where ξ is a divisor corresponding to the tautological line bundle.

The fan Δ of X admits the following explicit description. Let $\{e_1, \ldots, e_k\}$ be the canonical basis of $\mathbb{R}^k$, and set $e_0 := -e_1 - \ldots - e_k$. We also denote by e_j the vector $(0, e_j) \in \mathbb{R}^m \times \mathbb{R}^k$. Similarly, we use the same symbol η_i to denote the vector $(\eta_i, 0) \in \mathbb{R}^m \times \mathbb{R}^k$. For each $\eta_i \in \Sigma(1)$, set

$$\tilde{\eta}_i = \eta_i + \sum_{j=0}^{k} (a_{ij} - a_{i0}) e_j \in \mathbb{Z}^m \times \mathbb{Z}^k.$$

Then $\Delta(1) = \{e_0, \cdots, e_k, \tilde{\eta}_1, \cdots, \tilde{\eta}_r\}$, and the facet presentation of $P_0 * \ldots * P_k$ is given by:

$$\langle x, \tilde{\eta}_i\rangle \geq -a_{i0}, \ \ \langle x, e_0\rangle \geq -1, \ \ \langle x, e_j\rangle \geq 0,\ j = 1, \ldots, k.$$

For each cone $\sigma = \langle \eta_{i_1}, \ldots, \eta_{i_t}\rangle \in \Sigma(m)$, set $\tilde{\sigma} = \langle \tilde{\eta}_{i_1}, \ldots, \tilde{\eta}_{i_t}\rangle$. The maximal cones of Δ are of the form $\tilde{\sigma} + \langle e_0, \ldots, \hat{e_j}, \ldots, e_k\rangle$, for $\sigma \in \Sigma(m)$ and $j \in \{0, \ldots, k\}$.

The $\mathbb{P}^k$-bundle map $\pi : X \to Y$ is induced by the projection $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, and $\pi^*\big(V(\eta_i)\big) = V(\tilde{\eta}_i)$. Thus $L = V(e_0) + \sum_i a_{i0} V(\tilde{\eta}_i) = V(e_0) + \pi^*(D_0)$.

Next we consider the strict generalized Cayley polytope $\mathrm{Cayley}^s(P_0, \ldots, P_k)$, and the corresponding projective toric variety X. In [DDRP09], it was shown that there exists a toric fibration $\pi : X \to Y$ whose set theoretical fibers are all isomorphic to $\mathbb{P}^k$. We note that π may have multiple fibers, and X may be singular, even when Y is smooth. The following lemma gives a necessary and sufficient condition for $\mathrm{Cayley}^s(P_0, \ldots, P_k)$ to be smooth.

Lemma 16. *The polytope* $\mathrm{Cayley}^s(P_0, \ldots, P_k)$ *is smooth if and only if* Y *is smooth and* s *divides* $a_{ij} - a_{i0}$ *for every* $i \in \{1, \cdots, r\}$ *and* $j \in \{1, \cdots, k\}$. *In this case,* s *divides* $D_j - D_0$ *in* $\mathrm{Div}(Y)$ *for every* $j \in \{1, \cdots, k\}$, *and the corresponding polarized toric variety* (X, L) *satisfies:*

$$\begin{aligned} X \ &\cong \ \mathbb{P}_Y\Big(\mathcal{O}(D_0) \oplus \mathcal{O}\Big(\frac{D_1 - D_0}{s} + D_0\Big) \oplus \cdots \oplus \mathcal{O}\Big(\frac{D_k - D_0}{s} + D_0\Big)\Big)\ , \\ L \ &\sim \ s\xi + \pi^*\big((1-s)D_0\big)\ , \end{aligned}$$

where $\pi : X \to Y$ *is the* $\mathbb{P}^k$*-bundle map, and* ξ *is a divisor corresponding to the tautological line bundle.*

Proof. Set $P^1 := P_0 * ... * P_k$, and $P^s := \mathrm{Cayley}^s(P_0,\dots,P_k)$. Notice that a point $x=(y,z)\in\mathbb{R}^m\times\mathbb{R}^k$ lies in P^s if and only if $(y,\frac{z}{s})$ lies in P^1. Hence, from the facet description of P^1 given in paragraph 15, we deduce that P^s has the following facet presentation:

$$\langle x,\hat{\eta}_i\rangle \ge -a_{i0}, \quad \langle x,e_0\rangle \ge -s, \quad \langle x,e_j\rangle \ge 0 \quad j=1,...,k,$$

where $\hat{\eta}_i = \eta_i + \sum_{j=1}^{k}\frac{(a_{ij}-a_{i0})}{s}e_j$.

Suppose that Y is smooth and s divides $a_{ij}-a_{i0}$ for every $i\in\{1,\cdots,r\}$ and $j\in\{1,\cdots,k\}$. Then $\hat{\eta}_i$ is a primitive lattice vector, s divides D_j-D_0 in $\mathrm{Div}(Y)$, and one can check that P^s and $P_{D_0} * P_{\left(\frac{D_1-D_0}{s}+D_0\right)} * \cdots * P_{\left(\frac{D_k-D_0}{s}+D_0\right)}$ have the same normal fan. Thus

$$X \;\cong\; \mathbb{P}_Y\Big(\mathcal{O}(D_0)\oplus\mathcal{O}\Big(\frac{D_1-D_0}{s}+D_0\Big)\oplus\cdots\oplus\mathcal{O}\Big(\frac{D_k-D_0}{s}+D_0\Big)\Big),$$

and P^s is smooth. The facet presentations of P^s and $P_{D_0} * P_{\left(\frac{D_1-D_0}{s}+D_0\right)} * \cdots * P_{\left(\frac{D_k-D_0}{s}+D_0\right)}$ also show that

$$L \;=\; sV(e_0)+\sum a_{i0}V(\hat{\eta}_i) \;=\; sV(e_0)+\pi^*(D_0) \;\sim s\xi+\pi^*((1-s)D_0).$$

Conversely, suppose that P^s is smooth, and denote by Δ its normal fan. For each $i\in\{1,\cdots,r\}$, let r_i be the least positive (integer) number such that $r_i\hat{\eta}_i$ is a lattice vector. It follows from paragraph 15 that the maximal cones of Δ are of the form:

$$\langle r_{i_1}\hat{\eta}_{i_1},\dots,r_{i_t}\hat{\eta}_{i_t},e_0,\dots\hat{e}_j,\dots e_k\rangle,$$

where $\langle\eta_{i_1},\dots,\eta_{i_t}\rangle\in\Sigma(m)$ and $j\in\{0,\cdots,k\}$. Since X is smooth, Σ must be simplicial (i.e., $t=m$), and

$$1 \;=\; \big|det[r_{i_1}\hat{\eta}_{i_1},\dots,r_{i_m}\hat{\eta}_{i_m},e_0,\dots\hat{e}_j,\dots,e_k]\big| \;=\; \big|r_{i_1}\dots r_{i_m}\big|\cdot\big|det[\eta_{i_1},\dots,\eta_{i_m}]\big|.$$

It follows that $\big|det[\eta_{i_1},\dots,\eta_{i_m}]\big|=1$ for every $\langle\eta_{i_1},\dots,\eta_{i_m}\rangle\in\Sigma(m)$, and $r_i=1$ for every $i\in\{1,\cdots,r\}$. Thus Y is smooth, and s divides $a_{ij}-a_{i0}$ for every $i\in\{1,\cdots,r\}$ and $j\in\{1,\cdots,k\}$. $\square$

Next we give a sufficient condition for a generalized Cayley polytope to be $\mathbb{Q}$-normal, improving the criterion given in [DDRP09, Proposition 3.9].

Proposition 17. *Suppose that $P^s := \mathrm{Cayley}^s(P_0,\dots,P_k)$ is smooth, and $\frac{k+1}{s}\ge m$. Then one of the following conditions holds:*

(1) *P^s is $\mathbb{Q}$-normal and* $\mathrm{codeg}_{\mathbb{Q}}(P^s)=\frac{k+1}{s}$.

(2) *$Y\cong\mathbb{P}^m$, $P_i=d_i\Delta_m$ for positive integers d_i's such that $s\big|(d_i-d_0)$ for every i, and $sm\le\sum d_i< s(m+1)$. Up to renumbering, we may assume that $d_0\le d_1\le\cdots\le d_k$. There are 2 cases:*

  (a) *If $d_0=d_k$, then $P^s\cong d_0\Delta_m\times s\Delta_k$ is $\mathbb{Q}$-normal and* $\mathrm{codeg}_{\mathbb{Q}}(P^s)=\frac{m+1}{d_0}$.

  (b) *If $d_0<d_k$, then P^s is not $\mathbb{Q}$-normal,*

$$\tau(P^s)=\frac{k+1}{s}+\frac{m+1-\frac{\sum d_i}{s}}{d_0} \quad \textit{and} \quad \mathrm{codeg}_{\mathbb{Q}}(P^s)=\frac{k+1}{s}+\frac{m+1-\frac{\sum d_i}{s}}{d_k}.$$

Proof. The second projection $f:\mathbb{R}^m\times\mathbb{R}^k\to\mathbb{R}^k$ maps P^s onto the k-dimensional simplex $s\Delta_k$. Therefore, $\tau(P^s)\ge\mathrm{codeg}_{\mathbb{Q}}(P^s)\ge\mathrm{codeg}_{\mathbb{Q}}(\Delta_k)=\frac{k+1}{s}$.

Recall from 15 that the polarized toric variety associated to P^s is

$$(X,L) \;\cong\; \Big(\mathbb{P}_Y\big(\mathcal{O}(E_0)\oplus\cdots\oplus\mathcal{O}(E_k)\big),\; s\xi+\pi^*\big((1-s)D_0\big)\Big),$$

where $\pi : X \to Y$ is the $\mathbb{P}^k$-bundle map, ξ is a divisor corresponding to the tautological line bundle, and $E_i = \frac{D_i+(s-1)D_0}{s} \in \mathrm{Div}(X)$ for $i \in \{0,...,k\}$. We have $K_X \sim \pi^*(K_Y + E_0 + \ldots + E_k) - (k+1)\xi$. Since the D_i's are ample, the $\mathbb{Q}$-divisor

$$M \ := \ \sum_{i=0}^{k} E_i - \frac{(k+1)(s-1)}{s} D_0 \ = \ \frac{1}{s}\sum_{i=0}^{k} D_i$$

satisfies $M \cdot C \geq \frac{k+1}{s} \geq m$ for every invariant curve $C \subset Y$. By Remark 13, either $K_Y + M$ is nef, or $Y \cong \mathbb{P}^m$ and $sm \leq \sum_{i=0}^{k} d_i < s(m+1)$, where d_i denotes the degree of the ample divisor D_i under the isomorphism $Y \cong \mathbb{P}^m$.

Suppose $K_Y + M$ is nef. Then $\pi^*(K_Y + M)$ is nef but not ample on X. Since L is ample, it follows that

$$K_X + tL \ \sim \ \pi^*(K_Y + M) + \Big(t - \frac{k+1}{s}\Big)L$$

is ample if and only if $t > \frac{k+1}{s}$. Hence $\tau(P^s) = \frac{k+1}{s}$, as desired.

Suppose now $Y \cong \mathbb{P}^m$ and $sm \leq \sum_{i=0}^{k} d_i < s(m+1)$, and assume that $d_0 \leq d_1 \leq \cdots \leq d_k$. Then $X \cong \mathbb{P}_{\mathbb{P}^m}\big(\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_k)\big)$, where $a_i = d_0 + \frac{d_i - d_0}{s} \in \mathbb{Z}$. Using the notation of paragraph 15, one can check that

$$Nef(X) = \ cone\big([V(e_0)], [\pi^*H]\big) \quad and \quad \overline{Eff}(X) = \ cone\big([V(e_k)], [\pi^*H]\big),$$

where H is a hyperplane in $\mathbb{P}^m$. An easy computation then shows that

$$\tau(P^s) = \lambda(X,L)^{-1} = \frac{k+1}{s} + \frac{m+1-\frac{\sum d_i}{s}}{d_0} \ \text{ and } \ \mathrm{codeg}_{\mathbb{Q}}(P^s) = \sigma(X,L)^{-1} = \frac{k+1}{s} + \frac{m+1-\frac{\sum d_i}{s}}{d_k}.$$

We leave the details to the reader. $\square$

We now give an example of a (non $\mathbb{Q}$-normal) smooth lattice polytope P that satisfies $\mathrm{codeg}(P) = \frac{\dim(P)+1}{2}$ but P is not a generalized strict Cayley polytope.

Example 18. Let m be a positive integer, and $H \subset \mathbb{P}^m$ a hyperplane. Let $\pi : X \to \mathbb{P}^m \times \mathbb{P}^1$ be the blowup of $\mathbb{P}^m \times \mathbb{P}^1$ along $H_o := H \times \{o\}$. Then X is a smooth projective toric variety with Picard number 3. We will see below that X is Fano, and the Mori cone $\overline{NE}(X)$ has exacly 3 extremal rays, whose corresponding contractions are all divisorial contractions. Since X does not admit any contraction of fiber type, P_L is not a generalized strict Cayley polytope for any ample divisor L on X by Lemma 16. When m is even, we will then exhibit an ample divisor L on X such that $\mathrm{codeg}(P_L) = \frac{\dim(P_L)+1}{2}$.

Let $\{e_1, \ldots, e_m, e\}$ be the canonical basis of $\mathbb{R}^m \times \mathbb{R}$. The maximal cones of the fan Σ of $\mathbb{P}^m \times \mathbb{P}^1$ are of the form $\langle e_0, \ldots, \hat{e_i}, \ldots e_m, \pm e\rangle$, $i = 0, \ldots, m$. Set $f := e_1 + e$. The fan Σ_X of X is obtained from Σ by star subdivision centered in f. Set $D_i := V(e_i),\ i = 0, \ldots, m,\ D_e := V(e)$ and $E := V(f)$. One can check that $D_i \sim D_0$ for $i > 1$, $V(-e) \sim D_e + E$, and $D_0 \sim D_1 + E$. Therefore the classes of D_1, D_e and E form a basis for $N^1(X)$ and generate the cone $\overline{Eff}(X)$. Written in this basis,

$$-K_X = \sum_{i=0}^{m} D_i + D_e + V(-e) + E \sim (m+1)D_1 + 2D_e + (m+2)E.$$

The cone $\overline{NE}(X)$ is generated by the classes of the invariant curves C_1, C_2 and C_3 associated to the cones $\langle e_1, e_2, \dots e_m\rangle$, $\langle e_2, \dots e_m, f\rangle$ and $\langle e_2, \dots e_m, e\rangle$, respectively. For each $i \in \{1,2,3\}$, denote by π_i the contraction of the extremal ray generated by $[C_i]$. Then $\pi_1 : X \to \mathbb{P}_{\mathbb{P}^m}(\mathcal{O}(1) \oplus \mathcal{O})$ blows down the divisor D_1 onto a $\mathbb{P}^{m-1}$, $\pi_2 = \pi$, and $\pi_3 : X \to \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}^{\oplus m})$ blows down the divisor D_e onto a point.

In terms of the basis for $N^1(X)$ and $N_1(X)$ given above, the intersection product between divisors and curves is given by:

$$D_1 \cdot C_1 = -1, D_1 \cdot C_2 = 1, D_1 \cdot C_3 = 0$$

$$D_e \cdot C_1 = 0, D_e \cdot C_2 = 1, D_e \cdot C_3 = -1$$

$$E \cdot C_1 = 1, E \cdot C_2 = -1, E \cdot C_3 = 1.$$

By Kleiman's Ampleness Criterion, a divisor $D = aD_1 + bD_e + cE$ is ample if and only if $-a + c > 0$, $a + b - c > 0$ and $-b + c > 0$. Thus $L = 2D_1 + 2D_e + 3E$ is ample, and $K_X + tL$ is ample if and only if $t > m$. Hence $\tau(L) = m$. Since $\overline{Eff}(X) = \mathrm{Cone}\big([D_1], [D_e], [E]\big)$, $K_X + tL \in \overline{Eff}(X)$ if and only if $t \geq \frac{m+1}{2}$. Thus $\mathrm{codeg}_{\mathbb{Q}}(P_L) = \frac{m+1}{2}$. When m is even, $\mathrm{codeg}(P_L) = \lceil \mathrm{codeg}_{\mathbb{Q}}(P_L) \rceil = \frac{m+2}{2} = \frac{\dim(P_L)+1}{2}$.

Remark 19. In [Ito12], Ito characterized (not necessarily strict) Cayley polytopes of the form $P_0 * \dots * P_k$. They are the lattice polytopes whose corresponding polarized toric varieties are covered by k-planes.

In [Nob12], Nobili investigated a further generalization of strict Cayley polytopes, called *Cayley-Mori polytopes*. These are polytopes of the form $\mathrm{Conv}\big(P_0 \times \{0\}, P_1 \times \{w_1\}, \dots, P_k \times \{w_k\}\big) \subset \mathbb{R}^m \times \mathbb{R}^k$, where $P_0, \dots, P_k \subset \mathbb{R}^m$ are m-dimensional lattice polytopes with the same normal fan, and $w_1, \dots, w_k$ are lattice vectors that form a basis for $\mathbb{R}^k$. They are special cases of *twisted Cayley sums*, introduced by Casagrande and Di Rocco in [CDR08], and are precisely the lattice polytopes whose corresponding toric varieties are Mori fiber spaces.

4. Proof of Theorem 3

First note that the five classes of polytopes listed in Theorem 3 are $\mathbb{Q}$-normal and have codegree $\geq \frac{n+1}{2}$. This is straightforward for types (i), (ii) and (iii). For types (iv) and (v), this follows from Proposition 17.

Conversely, suppose that $P \subset \mathbb{R}^n$ is a smooth n-dimensional $\mathbb{Q}$-normal lattice polytope with $\mathrm{codeg}(P) \geq \frac{n+1}{2}$, and denote by (X, L) the corresponding polarized toric variety. We may assume that $n > 1$. By Remark 8, $\tau := \tau(P) > \frac{n-1}{2}$. Recall from Section 2.2 that $\tau = \lambda(X, L)^{-1}$. It follows from the discussion in Section 2.3 that the nef value morphism $\phi = \phi_L : X \to Y$ is defined by the linear system $\big|k(K_X + \tau L)\big|$ for k sufficiently large and divisible. Moreover, the assumption that P is $\mathbb{Q}$-normal implies that $\dim(Y) < \dim(X)$. If $C \subset X$ is an extremal curve contracted by ϕ, then

$$n + 1 \geq \mathfrak{l}\big(\mathbb{R}_+[C]\big) = -K_X \cdot C = \tau(L \cdot C) > \frac{n-1}{2} L \cdot C. \qquad (2)$$

In particular, $L \cdot C \leq 5$. We consider three cases:

Case 1. Suppose that $L \cdot C = 1$ for every extremal curve $C \subset X$ contracted by ϕ. Then $\tau = \mathrm{codeg}_{\mathbb{Q}}(P) = -K_X \cdot C \in \mathbb{Z}$, and thus $\tau = \mathrm{codeg}(P) \geq \frac{n+1}{2}$.

If $\dim(Y) = 0$, then $-K_X \sim \tau L$ is ample, i.e, X is a Fano manifold with index $r \geq \tau \geq \frac{n+1}{2}$. The classification in Section 2.4 implies that X is isomorphic to one of the following: $\mathbb{P}^n$, $\mathbb{P}^{\frac{n}{2}} \times \mathbb{P}^{\frac{n}{2}}$ (n even), $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ($n = 3$), or $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{r-1})$ ($n = 2r - 1$). In the first three cases we have $P \simeq \Delta_n$, $P \simeq \Delta_{\frac{n}{2}} \times \Delta_{\frac{n}{2}}$, and $P \simeq \Delta_1 \times \Delta_1 \times \Delta_1$ respectively. These are strict Cayley polytopes as in (iv). In the last case, let

$\pi : X \to \mathbb{P}^r$ be the $\mathbb{P}^{r-1}$-bundle map, and ξ a divisor on X corresponding to the tautological line bundle. One computes that $-K_X \sim r\xi$, and thus $L \sim \xi$. It follows from paragraph 15 that $P \simeq \mathrm{Cayley}^1(\underbrace{\Delta_r, \dots, \Delta_r}_{r-1\ times}, 2\Delta_r)$.

Suppose now that $\dim(Y) > 0$, and denote by X_y the general fiber of ϕ. By Theorem 11, applied in the toric context, there exists an extremal ray R of $\overline{NE}(X)$ whose associated contraction $\phi_R : X \to Z$ factors ϕ and realizes X as the projectivization of a vector bundle $\mathcal{E}$ of rank τ over a smooth toric variety $Z = X_\Sigma$. Set $k := \tau - 1$, let $F \simeq \mathbb{P}^k$ be a general fiber of ϕ_R, and $C \subset F$ a line. Since $L \cdot C = 1$, we have $\mathcal{O}_X(L)|_F \simeq \mathcal{O}_{\mathbb{P}^k}(1)$. By Fujita's Lemma (see for instance [BS95, 3.2.1]), $X \simeq_Z \mathbb{P}_Z(\phi_{R*}\mathcal{O}_X(L))$, and under this isomorphism $\mathcal{O}_X(L)$ corresponds to the tautological line bundle. Since X is toric, the ample vector bundle $\phi_{R*}\mathcal{O}_X(L)$ splits as a sum of $k+1$ ample line bundles on Z. It follows from paragraph 15 that there are polytopes $P_0, \dots, P_k$ with normal fan Σ such that $P \simeq \mathrm{Cayley}^1(P_0, \dots, P_k)$. Note moreover that $k = \tau - 1 \geq \frac{n-1}{2}$.

Case 2. Suppose that there is an extremal curve $C \subset X$ contracted by ϕ such that $L \cdot C = 2$. Let R be the extremal ray generated by C, and $\phi_R : X \to Z$ the associated contraction. By (2), $\mathfrak{l}(R) = -K_X \cdot C \in \{n, n+1\}$. Let E be the excepcional locus of ϕ_R, and F an irreducible component of a fiber of the restriction $\phi_R|_E$. By (1), $\dim(E) = n$ and $n - 1 \leq \dim(F) \leq n$.

If $\dim(F) = n$, then $(X, \mathcal{O}_X(L)) \simeq (\mathbb{P}^n, \mathcal{O}(2))$, and $P \simeq 2\Delta_n$.

If $\dim(F) = n-1$, then $\phi_R : X \to \mathbb{P}^1$ is a $\mathbb{P}^{n-1}$-bundle, as explained in Section 2.3, and $\mathcal{O}_X(L)|_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2)$. So there are integers $0 < a_0 \leq \cdots \leq a_{n-1}$ and $a > -2a_0$ such that

$$\begin{aligned} X &\cong \mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_{n-1})\big) \ , \\ L &\sim 2\xi + aF \ , \end{aligned}$$

where ξ a divisor corresponding to the tautological line bundle. By Lemma 16,

$$P \cong \mathrm{Cayley}^2\left((2a_0 - a)\Delta_1, \dots, (2a_{n-1} - a)\Delta_1\right).$$

Case 3. Suppose that there is an extremal curve $C \subset X$ contracted by ϕ such that $3 \leq L \cdot C \leq 5$. By (2), we must have $n \leq 4$. If $3 \leq n \leq 4$, then (1) and (2) imply that $L \cdot C = 3$ and $X \cong \mathbb{P}^n$. Thus $P \cong 3\Delta_n$. For $n \in \{3, 4\}$, $\mathrm{codeg}(3\Delta_n) = 2$. This is $\geq \frac{n+1}{2}$ only if $n = 3$.

From now on suppose $n = 2$. If $L \cdot C \in \{4, 5\}$, then (1) implies that $-K_X \cdot C = 3$, and thus $X \cong \mathbb{P}^2$. On the other hand, $4\Delta_2$ and $5\Delta_2$ do not satisfy the codegree hypothesis. So we must have $L \cdot C = 3$. We conclude from (1) and (2) that there are integers $0 < a_0 \leq a_1$ and $a > -3a_0$ such that

$$\begin{aligned} X &\cong \mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}(a_0) \oplus \mathcal{O}(a_1)\big) \ , \\ L &\sim 3\xi + aF \ , \end{aligned}$$

where ξ a divisor corresponding to the tautological line bundle, and F is a fiber of $X \to \mathbb{P}^1$. By Lemma 16,

$$P \cong \mathrm{Cayley}^3\left((3a_0 - a)\Delta_1, (3a_1 - a)\Delta_1\right).$$

On the other hand, the latter has codegree $= 1 < \frac{n+1}{2}$. So this case does not occur. □

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